

---

# Control of Networked Multiagent Systems with Uncertain Graph Topologies<sup>†</sup>

---

**Tansel Yucelen\*** and **John Daniel Peterson**

*Department of Mechanical and Aerospace Engineering  
Missouri University of Science and Technology*

**Kevin L. Moore**

*Department of Electrical Engineering and Computer Science  
Colorado School of Mines*

**Abstract** — Multiagent systems consist of agents that locally exchange information through a physical network subject to a graph topology. Current control methods for networked multiagent systems assume the knowledge of graph topologies in order to design distributed control laws for achieving desired global system behaviors. However, this assumption may not be valid for situations where graph topologies are subject to uncertainties either due to changes in the physical network or the presence of modeling errors especially for multiagent systems involving a large number of interacting agents. Motivating from this standpoint, this paper studies distributed control of networked multiagent systems with uncertain graph topologies. The proposed framework involves a controller architecture that has an ability to adapt its feedback gains in response to system variations. Specifically, we analytically show that the proposed controller drives the trajectories of a networked multiagent system subject to a graph topology with time-varying uncertainties to a close neighborhood of the trajectories of a given reference model having a desired graph topology. As a special case, we also show that a networked multiagent system subject to a graph topology with constant uncertainties asymptotically converges to the trajectories of a given reference model. Although the main result of this paper is presented in the context of average consensus problem, the proposed framework can be used for many other problems related to networked multiagent systems with uncertain graph topologies.

**Keywords** — Networked multiagent systems; uncertain graph topologies; distributed control; adaptive control; stability

---

<sup>†</sup> This research was supported in part by the University of Missouri Research Board.

\* Corresponding author: 400 West, 13th Street, Rolla, MO 65409 (Address); +1 573 341 7702 (Phone); +1 573 341 6899 (Fax); yucelen@mst.edu (Email).

# 1. Introduction

Multiagent systems consist of agents that locally exchange information through a physical network subject to a graph topology. Current control methods for networked multiagent systems assume the knowledge of graph topologies in order to design distributed control laws for achieving desired global system behaviors (see, for example, [1] and references therein). However, this assumption may not be valid for situations where graph topologies are subject to uncertainties either due to changes in the physical network or the presence of modeling errors especially for multiagent systems involving a large number of interacting agents.

Uncertain nature of networked multiagent systems has received a considerable attention recently, including notable results [2–10]. For achieving desired multiagent system behavior, [2,3] make a specific assumption on the network connectivity other than the standard assumption on the connectedness of networked agents. The authors of [4] excite the multiagent system in order to detect and isolate the uncertain agents from the network topology. Like [2,3], a computationally expensive and not scalable algorithm is proposed in [5,6] based on input observers technique, where the effect of uncertain agents on the overall multiagent system performance is quantified. An extension of this work is also given in [7] that focuses on the detection and isolation of uncertain agents. The authors of [8,9] use an adaptive control approach in order to suppress the effect of uncertain agents on the overall multiagent system performance without making specific assumptions on the fraction of misbehaving agents. A common similarity of the approaches documented in [2–9] is that they model uncertainties in the agent dynamics as additive perturbations that do not depend on the state of agents, where these results are not applicable to the networked multiagent systems with graph topology uncertainties since such uncertainties depend on the state of agents.

One relevant work to the results of this paper is recently appeared in [10], where the authors utilize adaptive and sliding mode control methodologies in order to enforce a networked multiagent system subject to an uncertain graph topology to follow a given reference model having a desired graph topology. However, the result in [10] may require a centralized information

exchange among networked agents in general due to the structure of the proposed control algorithm (see (8) or (18) of [10]). Other important results, which are related to this paper, are presented in [11–13] without requiring a centralized information exchange. However, these results hold for graph topologies subject to constant uncertainties only.

In this paper, we study distributed control of networked multiagent systems with uncertain graph topologies. The proposed framework involves a novel controller architecture that has an ability to adapt its feedback gains in response to system variations. Specifically, we analytically show that the proposed controller drives the trajectories of a networked multiagent system subject to a graph topology with time-varying uncertainties to a close neighborhood of the trajectories of a given reference model having a desired graph topology. As a special case, we also show that a networked multiagent system subject to a graph topology with constant uncertainties asymptotically converges to the trajectories of a given reference model. Although the main result of this paper is presented in the context of average consensus problem, the proposed framework can be used for many other problems related to networked multiagent systems with uncertain graph topologies.

## 2. Notation, Definitions, and Graph-Theoretic Notions

The notation used in this paper is fairly standard. Specifically,  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}^n$  denotes the set of  $n \times 1$  real column vectors,  $\mathbb{R}^{n \times m}$  denotes the set of  $n \times m$  real matrices,  $\mathbb{R}_+$  denotes the set of positive real numbers,  $\mathbb{R}_+^{n \times n}$  (resp.,  $\overline{\mathbb{R}}_+^{n \times n}$ ) denotes the set of  $n \times n$  positive-definite (resp., nonnegative-definite) real matrices,  $\mathbb{S}_+^{n \times n}$  (resp.,  $\overline{\mathbb{S}}_+^{n \times n}$ ) denotes the set of  $n \times n$  symmetric positive-definite (resp., symmetric nonnegative-definite) real matrices,  $\mathbf{0}_n$  (resp.,  $\mathbf{1}_n$ ) denotes the  $n \times 1$  vector of all zeros (resp., ones), and  $I_n$  denotes the  $n \times n$  identity matrix. Furthermore, we write  $(\cdot)^T$  for transpose,  $\|\cdot\|_2$  for the Euclidian norm,  $\lambda_{\min}(A)$  (resp.,  $\lambda_{\max}(A)$ ) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix  $A$ ,  $\lambda_i(A)$  for the  $i$ -th eigenvalue of  $A$  ( $A$  is symmetric and the eigenvalues are ordered from least to greatest value),  $\text{diag}(a)$  for the diagonal matrix with the vector  $a$  on its diagonal, and  $[A]_{ij}$  for the entry of the matrix  $A$  on the  $i$ -th row and  $j$ -th column.

Next, we recall some of the basic notions from graph theory, where we refer to [1] for further details. In the multiagent literature, graphs are broadly adopted to encode interactions in networked systems. An *undirected* graph  $\mathcal{G}$  is defined by a set  $\mathcal{V}_{\mathcal{G}} = \{1, \dots, n\}$  of *nodes* and a set  $\mathcal{E}_{\mathcal{G}} \subset \mathcal{V}_{\mathcal{G}} \times \mathcal{V}_{\mathcal{G}}$  of *edges*. If  $(i, j) \in \mathcal{E}_{\mathcal{G}}$ , then the nodes  $i$  and  $j$  are *neighbors* and the neighboring relation is indicated with  $i \sim j$ . The *degree* of a node is given by the number of its neighbors. Letting  $d_i$  be the degree of node  $i$ , then the *degree* matrix of a graph  $\mathcal{G}$ ,  $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ , is given by  $\mathcal{D}(\mathcal{G}) \triangleq \text{diag}(d)$ ,  $d = [d_1, \dots, d_n]^T$ . A *path*  $i_0 i_1 \dots i_L$  is a finite sequence of nodes such that  $i_{k-1} \sim i_k$ ,  $k = 1, \dots, L$ , and a graph  $\mathcal{G}$  is *connected* if there is a path between any pair of distinct nodes. The *adjacency* matrix of a graph  $\mathcal{G}$ ,  $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n \times n}$ , is given by

$$[\mathcal{A}(\mathcal{G})]_{ij} \triangleq \begin{cases} 1, & \text{if } (i, j) \in \mathcal{E}_{\mathcal{G}}, \\ 0, & \text{otherwise.} \end{cases}$$

The *Laplacian* matrix of a graph,  $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$ , playing a central role in many graph theoretic treatments of multiagent systems, is given by  $\mathcal{L}(\mathcal{G}) \triangleq \mathcal{D}(\mathcal{G}) - \mathcal{A}(\mathcal{G})$ , where the spectrum of the Laplacian of a connected, undirected graph  $\mathcal{G}$  can be ordered as

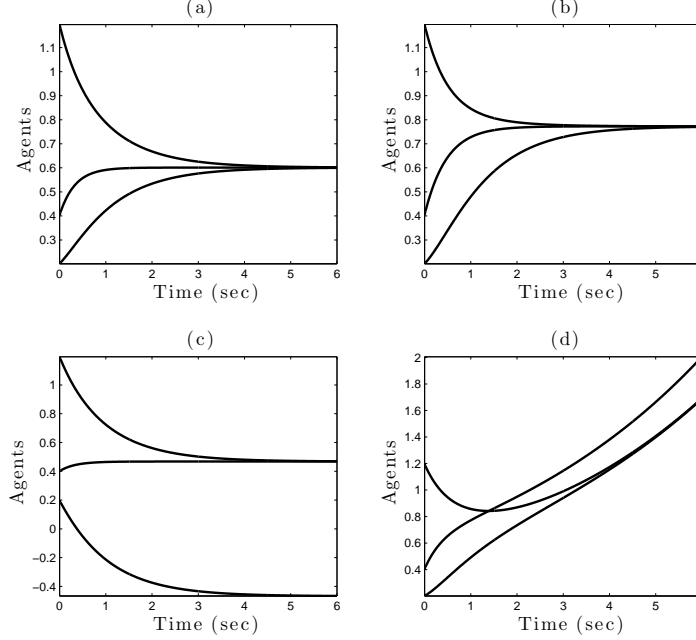
$$0 = \lambda_1(\mathcal{L}(\mathcal{G})) < \lambda_2(\mathcal{L}(\mathcal{G})) \leq \dots \leq \lambda_n(\mathcal{L}(\mathcal{G})), \quad (1)$$

with  $\mathbf{1}_n$  as the eigenvector corresponding to the zero eigenvalue  $\lambda_1(\mathcal{L}(\mathcal{G}))$  and  $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$  and  $e^{\mathcal{L}(\mathcal{G})}\mathbf{1}_n = \mathbf{1}_n$ .

### 3. Problem Formulation

Consider a multiagent system consisting of  $n$  agents that locally exchange information according to a connected, undirected *uncertain* graph  $\mathcal{G}_u$  with nodes and edges representing agents and interagent information exchange links, respectively. We assume that the network is static, and hence, agent evolution will not cause edges to appear or disappear in the network. Specifically, let  $x_i(t) \in \mathbb{R}$  denote the state of node  $i$  at time  $t \in \overline{\mathbb{R}}_+$ , whose dynamics is described by

$$\dot{x}_i(t) = -\alpha_i(t)x_i(t) + \sum_{i \sim j} \beta_{ij}(t)x_j(t) + u_i(t), \quad x_i(0) = x_{i0}, \quad (2)$$



**Figure 1:** Trajectories of three agents on a line graph subject to initial conditions  $(x_{10}, x_{20}, x_{30}) = (0.2, 0.4, 1.2)$  for (a)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 1, 1, 1)$ , (b)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 0.1, 1, 1)$ , (c)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (-1, -1, 1, 1)$ , and (d)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1.5, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 1, 1, 1)$ .

where  $\alpha_i(t) \in \mathbb{R}$  and  $\beta_{ij}(t) \in \mathbb{R}$  are unknown bounded coefficients of the graph  $\mathcal{G}_u$  with bounded time derivatives, and  $u_i(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ , is the control input of node  $i$ . In this paper, we are interested to design a distributed control input  $u_i(t)$ ,  $t \in \overline{\mathbb{R}}_+$ , such that (2) achieves average consensus approximately (or asymptotically, i.e.,  $x(t) \rightarrow (\mathbf{1}_n \mathbf{1}_n^T / n) x_0$  as  $t \rightarrow \infty$ ,  $x(t) = [x_1(t), \dots, x_n(t)]^T \in \mathbb{R}^n$ ) in the presence of an uncertain graph topology.

**Remark 1.** In the absence of proper control inputs  $u_i(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ , (2) cannot necessarily achieve average consensus. To elucidate this point, let unknown coefficients of the graph  $\mathcal{G}_u$  be constant, i.e.,  $(\alpha_i(t), \beta_{ij}(t)) = (\alpha_i, \beta_{ij})$ , and consider four cases given in Figure 1 that show trajectories of three agents on a line graph subject to initial conditions  $(x_{10}, x_{20}, x_{30}) = (0.2, 0.4, 1.2)$ . Since  $\alpha_i = \sum_{j \sim i} \beta_{ij}$  and  $\beta_{ij} = \beta_{ji}$  in case (a), this case results in average consensus at point  $(\mathbf{1}_n \mathbf{1}_n^T / n) x_0 = 0.6$ . Since  $\beta_{12} \neq \beta_{21}$  in case (b), this case does not result in average consensus at point 0.6. Case (c) considers a multiagent system with antagonistic interactions [14], and hence,

it does not result in average consensus due to the existence of multiple equilibrium points. Finally, since  $\alpha_i \neq \sum_{i \sim j} \beta_{ij}$  for  $i = 2$  in case (d), this case does not result in average consensus as well. In summary, (2) results in average consensus if  $\alpha_i = \sum_{i \sim j} \beta_{ij}$  and  $\beta_{ij} = \beta_{ji} \in \mathbb{R}_+$  [15]. However, this cannot be justified due to unknown coefficients of the graph  $\mathcal{G}_u$ , and hence, one needs to design proper control inputs  $u_i(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ .

Next, we propose a control input  $u_i(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ , to drive the trajectories of (2) to a close neighborhood of a given reference model having a desired graph topology without requiring a centralized information exchange among networked agents. For this purpose, consider the reference model that locally exchange information according to a connected, undirected graph  $\mathcal{G}$  given by

$$\dot{r}_i(t) = - \sum_{i \sim j} (r_i(t) - r_j(t)), \quad r_i(0) = x_{i0}, \quad (3)$$

where  $r_i(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ , denotes the state of the reference model for node  $i$ . Note that

$$\lim_{t \rightarrow \infty} r(t) = (\mathbf{1}_n \mathbf{1}_n^T / n) x_0, \quad (4)$$

where  $r(t) = [r_1(t), \dots, r_n(t)]^T \in \mathbb{R}^n$ . Throughout this paper we assume that the nodes and edges of graphs  $\mathcal{G}$  and  $\mathcal{G}_u$  coincide, however the graph  $\mathcal{G}_u$  is subject to unknown coefficients  $\alpha_i(t)$  and  $\beta_{ij}(t)$ , as discussed earlier.

**Remark 2.** The reference model given by (3) can be easily extended to

$$\dot{r}_i(t) = - \sum_{i \sim j} \xi_{ij} (r_i(t) - r_j(t)), \quad r_i(0) = x_{i0}, \quad (5)$$

without changing the following results of this paper, where  $\xi_{ii} = \sum_{i \sim j} \xi_{ij}$  and  $\xi_{ij} = \xi_{ji} \in \mathbb{R}_+$ .

**Remark 3.** Note that (3) can be equivalently written as

$$\dot{r}_i(t) = -d_i r_i(t) + \sum_{i \sim j} r_j(t), \quad r_i(0) = x_{i0}, \quad (6)$$

where  $d_i$  is the degree of node  $i$  on graph  $\mathcal{G}$ . Therefore, if one knows the coefficients  $\alpha_i(t)$  and  $\beta_{ij}(t)$  of 2, then the control input

$$u_i(t) = -(d_i - \alpha_i(t))x_i(t) + \sum_{i \sim j} (1 - \beta_{ij}(t))x_j(t), \quad (7)$$

results in average consensus at point  $(\mathbf{1}_n \mathbf{1}_n^T / n) x_0$ .

Since the control input (7) given in Remark 3 is not feasible due to unknown coefficients  $\alpha_i(t)$  and  $\beta_{ij}(t)$ , we propose the adaptive control input given by

$$u_i(t) = -k_i(x_i(t) - r_i(t)) - \hat{w}_i(t)x_i(t) - \sum_{i \sim j} \hat{w}_{ij}(t)x_j(t), \quad (8)$$

where  $k_i \in \mathbb{R}_+$  for at least one agent or a subset of agents (and  $k_i = 0$  for others), and the estimates  $\hat{w}_i(t) \in \mathbb{R}$  and  $\hat{w}_{ij}(t) \in \mathbb{R}$ ,  $t \in \overline{\mathbb{R}}_+$ , satisfy the update laws

$$\dot{\hat{w}}_i(t) = \gamma_i \text{Proj}\left(\hat{w}_i(t), x_i(t)(x_i(t) - r_i(t))\right), \quad \hat{w}_i(0) = \hat{w}_{i0}, \quad (9)$$

$$\dot{\hat{w}}_{ij}(t) = \gamma_{ij} \text{Proj}\left(\hat{w}_{ij}(t), x_j(t)(x_i(t) - r_i(t))\right), \quad \hat{w}_{ij}(0) = \hat{w}_{ij0}, \quad (10)$$

with  $\gamma_i \in \mathbb{R}_+$  and  $\gamma_{ij} \in \mathbb{R}_+$  being the corresponding learning rates. In the update laws given by (9) and (10), Proj denotes the projection operator [16, 17], which is used to keep the estimates  $\hat{w}_i(t)$  and  $\hat{w}_{ij}(t)$  bounded for all  $t \in \overline{\mathbb{R}}_+$ . In the next section, we analytically show that the proposed adaptive control input given by (8) along with the update laws (9) and (10) drives the trajectories of (2) to a close neighborhood of the reference model trajectories given by (5).

## 4. Stability Analysis

In this section, we establish stability properties of the proposed adaptive control input given by (8) along with the update laws (9) and (10). For this purpose, let

$$e_i(t) \triangleq x_i(t) - r_i(t), \quad t \in \overline{\mathbb{R}}_+, \quad (11)$$

denote the local error dynamics that satisfy

$$\begin{aligned} \dot{e}_i(t) &= -\alpha_i(t)x_i(t) + \sum_{i \sim j} \beta_{ij}(t)x_j(t) + u_i(t) + d_i r_i(t) - \sum_{i \sim j} r_j(t) \\ &= -\alpha_i(t)x_i(t) + \sum_{i \sim j} \beta_{ij}(t)x_j(t) + u_i(t) + d_i r_i(t) - \sum_{i \sim j} r_j(t) + d_i x_i(t) \\ &\quad - d_i x_i(t) + \sum_{i \sim j} x_j(t) - \sum_{i \sim j} x_j(t) \\ &= -d_i e_i(t) + \sum_{i \sim j} e_j(t) + (d_i - \alpha_i(t))x_i(t) + \sum_{i \sim j} (\beta_{ij}(t) - 1)x_j(t) + u_i(t) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{i \sim j} (e_i(t) - e_j(t)) + w_i(t)x_i(t) + \sum_{i \sim j} w_{ij}(t)x_j(t) + u_i(t) \\
&= - \sum_{i \sim j} (e_i(t) - e_j(t)) + w_i(t)x_i(t) + \sum_{i \sim j} w_{ij}(t)x_j(t) - k_i e_i(t) \\
&\quad - \hat{w}_i(t)x_i(t) - \sum_{i \sim j} \hat{w}_{ij}(t)x_j(t) \\
&= -k_i e_i(t) - \sum_{i \sim j} (e_i(t) - e_j(t)) - \tilde{w}_i(t)x_i(t) - \sum_{i \sim j} \tilde{w}_{ij}(t)x_j(t), \quad e_i(0) = 0,
\end{aligned} \tag{12}$$

where

$$\tilde{w}_i(t) \triangleq \hat{w}_i(t) - w_i(t), \quad t \in \overline{\mathbb{R}}_+, \tag{13}$$

$$\tilde{w}_{ij}(t) \triangleq \hat{w}_{ij}(t) - w_{ij}(t), \quad t \in \overline{\mathbb{R}}_+, \tag{14}$$

$w_i(t) \triangleq d_i - \alpha_i(t)$ ,  $t \in \overline{\mathbb{R}}_+$ , and  $w_{ij}(t) \triangleq \beta_{ij}(t) - 1$ ,  $t \in \overline{\mathbb{R}}_+$ . In addition, it follows from (9) and (10) that

$$\dot{\tilde{w}}_i(t) = \gamma_i \text{Proj}(\hat{w}_i(t), x_i(t)(x_i(t) - r_i(t))) - \dot{w}_i(t), \quad \tilde{w}_i(0) = \tilde{w}_{i0}, \tag{15}$$

$$\dot{\tilde{w}}_{ij}(t) = \gamma_{ij} \text{Proj}(\hat{w}_{ij}(t), x_j(t)(x_i(t) - r_i(t))) - \dot{w}_{ij}(t), \quad \tilde{w}_{ij}(0) = \tilde{w}_{ij0}, \tag{16}$$

where  $\tilde{w}_{i0} \triangleq \hat{w}_{i0} - w_i$  and  $\tilde{w}_{ij0} \triangleq \hat{w}_{ij0} - w_{ij}$ . Note that  $\dot{w}_i(t)$  and  $\dot{w}_{ij}(t)$  are bounded since it is assumed that unknown bounded coefficients  $\alpha_i(t)$  and  $\beta_{ij}(t)$  have bounded time derivatives.

We now state the following lemma necessary for the results of this section.

**Lemma 1.** Let  $K = \text{diag}(k)$ ,  $k = [k_1, k_2, \dots, k_n]^T$ ,  $k_i \in \overline{\mathbb{R}}_+$ ,  $i = 1, \dots, n$ , and assume that at least one element of  $k$  is nonzero. Then, for the Laplacian of a connected, undirected graph,

$$\mathcal{F}(\mathcal{G}) \triangleq \mathcal{L}(\mathcal{G}) + K \in \mathbb{S}_+^{n \times n}, \tag{17}$$

and  $\det(\mathcal{F}(\mathcal{G})) \neq 0$ .

*Proof.* Consider the decomposition  $K = K_1 + K_2$ , where  $K_1 \triangleq \text{diag}([0, \dots, 0, \phi_i, 0, \dots, 0]^T)$  and  $K_2 \triangleq K - K_1$ , where  $\phi_i$  denotes the smallest nonzero diagonal element of  $K$  appearing on its  $i$ -th diagonal, so that  $K_2 \in \overline{\mathbb{S}}_+^{n \times n}$ . From the Rayleigh's Quotient [18], the minimum eigenvalue of  $\mathcal{L}(\mathcal{G}) + K_1$  can be given by

$$\lambda_{\min}(\mathcal{L}(\mathcal{G}) + K_1) = \min_x \{x^T (\mathcal{L}(\mathcal{G}) + K_1) x \mid x^T x = 1\}, \tag{18}$$



where  $x$  is the eigenvector corresponding to this minimum eigenvalue. Note that since  $\mathcal{L}(\mathcal{G}) \in \overline{\mathbb{S}}_+^{n \times n}$  and  $K_1 \in \overline{\mathbb{S}}_+^{n \times n}$ , and hence,  $\mathcal{L}(\mathcal{G}) + K_1$  is real and symmetric,  $x$  is a real eigenvector. Now, expanding (18) as

$$x^T (\mathcal{L}(\mathcal{G}) + K_1) x = \sum_{i \sim j} (x_i - x_j)^2 + \phi_i x_i^2, \quad (19)$$

and noting that the right hand side of (19) is zero only if  $x \equiv 0$ , it follows that  $\lambda_{\min}(\mathcal{L}(\mathcal{G}) + K_1) > 0$ , and hence,  $\mathcal{L}(\mathcal{G}) + K_1 \in \mathbb{S}_+^{n \times n}$ . Finally, let  $\lambda$  be an eigenvalue of  $\mathcal{F}(\mathcal{G}) = \mathcal{L}(\mathcal{G}) + K_1 + K_2$ . Since  $\lambda_{\min}(\mathcal{L}(\mathcal{G}) + K_1) > 0$  and  $\lambda_{\min}(K_2) = 0$ , it follows from Fact 5.11.3 of [19] that  $\lambda_{\min}(\mathcal{L}(\mathcal{G}) + K_1) + \lambda_{\min}(K_2) \leq \lambda$ , and hence,  $\lambda > 0$ , which implies that (17) holds and  $\det(\mathcal{F}(\mathcal{G})) \neq 0$ . ■

The next theorem presents the first result of this section.

**Theorem 1.** Consider the networked multiagent system given by (2) subject to an uncertain graph topology, the reference model given by (3), the adaptive control input given by (8), and the update laws given by (9) and (10). Then, the solution  $(e_i(t), \tilde{w}_i(t), \tilde{w}_{ij}(t))$  of the closed-loop dynamical system given by (12), (15), and (16) is bounded for all  $(0, \tilde{w}_{i0}, \tilde{w}_{ij0})$ ,  $t \in \overline{\mathbb{R}}_+$ , and  $(i, j)$ .

*Proof.* First consider the quadratic function given by

$$V_i(e_i, \tilde{w}_i, \tilde{w}_{ij}) = \frac{1}{2} \left( e_i^2 + \gamma_i^{-1} \tilde{w}_i^2 + \sum_{i \sim j} \gamma_{ij}^{-1} \tilde{w}_{ij}^2 \right), \quad (20)$$

and note that  $V_i(0, 0, 0) = 0$  and  $V_i(e_i, \tilde{w}_i, \tilde{w}_{ij}) \in \mathbb{R}_+$ ,  $(e_i, \tilde{w}_i, \tilde{w}_{ij}) \neq (0, 0, 0)$ . Furthermore,  $V_i(e_i, \tilde{w}_i, \tilde{w}_{ij})$  is radially unbounded. Differentiating (20) along the closed-loop trajectories of (12), (15), and (16) yields

$$\dot{V}_i(e_i(t), \tilde{w}_i(t), \tilde{w}_{ij}(t)) \leq -e_i(t) \sum_{i \sim j} (e_i(t) - e_j(t)) - k_i e_i^2(t) + w_i^*, \quad (21)$$

where  $w_i^*$  is an upper bound satisfying  $\|\gamma_i^{-1}(\hat{w}_i(t) - w_i(t))\dot{w}_i(t) + \sum_{i \sim j} \gamma_{ij}^{-1}(\hat{w}_{ij}(t) - w_{ij}(t)) \cdot \dot{w}_{ij}(t)\|_2 \leq w_i^*$ ,  $t \in \overline{\mathbb{R}}_+$ . Note that  $w_i^*$  exists since all the terms inside the norm operator are bounded and projection operator is used for the estimates  $\hat{w}_i(t)$  and  $\hat{w}_{ij}(t)$ . Now, consider the Lyapunov function candidate given by

$$V(\cdot) = \sum_{i=1}^n V_i(e_i, \tilde{w}_i, \tilde{w}_{ij}). \quad (22)$$

The time derivative of (22) is given using (21) as

$$\dot{V}(\cdot) \leq -e^T(t)(\mathcal{L}(\mathcal{G}) + K)e(t) + w^*, \quad w^* \triangleq \sum_{i=1}^n w_i^*, \quad (23)$$

where  $\mathcal{L}(\mathcal{G})$  denotes the Laplacian matrix of (3),  $K \triangleq \text{diag}(k)$ ,  $k = [k_1, k_2, \dots, k_n]^T$ ,  $k_i \in \overline{\mathbb{R}}_+$ , and  $e(t) = [e_1(t), \dots, e_n(t)]^T$ . From the definition of the adaptive control input in (8), notice that at least one element of  $k$  is nonzero. This implies from Lemma 1 that  $\mathcal{L}(\mathcal{G}) + K \in \mathbb{S}_+^{n \times n}$  and  $\det(\mathcal{L}(\mathcal{G}) + K) \neq 0$ , and hence,  $\lambda_{\min}(\mathcal{L}(\mathcal{G}) + K) \|e(t)\|_2 \leq e^T(t)(\mathcal{L}(\mathcal{G}) + K)e(t)$ . Now, since  $\dot{V}(\cdot) \leq 0$  when  $\|e(t)\|_2 \geq w^* / \lambda_{\min}(\mathcal{L}(\mathcal{G}) + K)$ , it follows that the closed-loop dynamical system given by (12), (15), and (16) is bounded for all  $(0, \tilde{w}_{i0}, \tilde{w}_{ij0})$ ,  $t \in \overline{\mathbb{R}}_+$ , and  $(i, j)$ . ■

**Remark 4.** In order to drive the trajectories of (2) to a close neighborhood of the reference model trajectories given by (5), the perturbation term  $w^*$  in (25) needs to be small. This holds if the time derivative of unknown coefficients  $\alpha_i(t)$  and  $\beta_{ij}(t)$  is small. If this is not true, then one can increase the learning rates  $\gamma_i$  and  $\gamma_{ij}$  to make  $w^*$  small.

As a special case when the unknown coefficients are constant, i.e.,  $(\alpha_i(t), \beta_{ij}(t)) = (\alpha_i, \beta_{ij})$ , the next theorem shows that the proposed adaptive control input given by (8) along with the update laws (9) and (10) asymptotically drives the trajectories of (2) to the reference model trajectories given by (5).

**Theorem 2.** Consider the networked multiagent system given by (2) subject to an uncertain graph topology, the reference model given by (3), the adaptive control input given by (8), and the update laws given by (9) and (10). Then, the solution  $(e_i(t), \tilde{w}_i(t), \tilde{w}_{ij}(t))$  of the closed-loop dynamical system given by (12), (15), and (16) is Lyapunov stable for all  $(0, \tilde{w}_{i0}, \tilde{w}_{ij0})$ ,  $t \in \overline{\mathbb{R}}_+$ , and  $(i, j)$ , and  $\lim_{t \rightarrow \infty} e_i(t) = 0$  for all  $i$ . In addition,  $\lim_{t \rightarrow \infty} x(t) = (\mathbf{1}_n \mathbf{1}_n^T / n) x_0$ .

*Proof.* To show the Lyapunov stability of the closed-loop dynamical system given by (12), (15), and (16), first consider the quadratic function given by (20). Differentiating (20) along the closed-loop trajectories of (12), (15), and (16) yields

$$\dot{V}_i(e_i(t), \tilde{w}_i(t), \tilde{w}_{ij}(t)) \leq -e_i(t) \sum_{i \sim j} (e_i(t) - e_j(t)) - k_i e_i^2(t). \quad (24)$$

Now, consider the Lyapunov function candidate given by (22), where the time derivative of (22)

is given using (24) as

$$\dot{V}(\cdot) \leq -e^T(t)(\mathcal{L}(\mathcal{G}) + K)e(t), \quad (25)$$

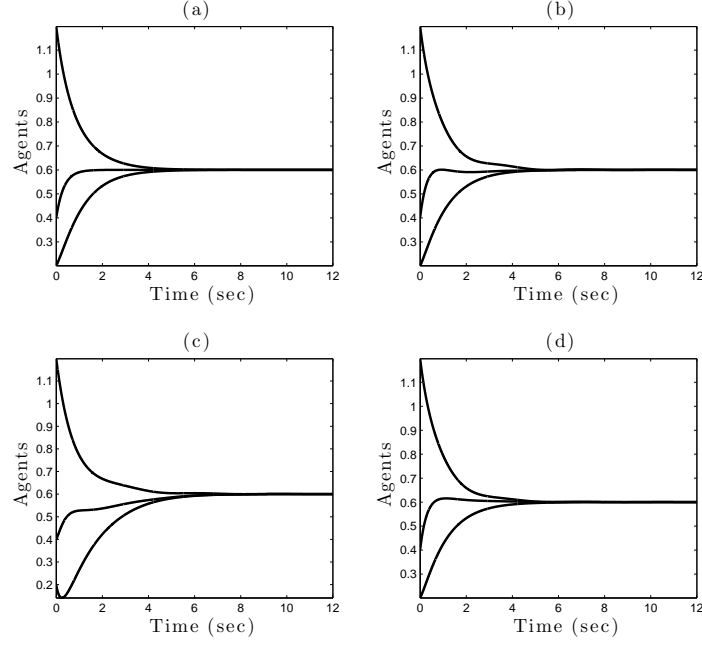
where  $\mathcal{L}(\mathcal{G})$  denotes the Laplacian matrix of (3),  $K \triangleq \text{diag}(k)$ ,  $k = [k_1, k_2, \dots, k_n]^T$ ,  $k_i \in \overline{\mathbb{R}}_+$ , and  $e(t) = [e_1(t), \dots, e_n(t)]^T$ . From the definition of the adaptive control input in (8), notice that at least one element of  $k$  is nonzero. This implies from Lemma 1 that  $\mathcal{L}(\mathcal{G}) + K \in \mathbb{S}_+^{n \times n}$  and  $\det(\mathcal{L}(\mathcal{G}) + K) \neq 0$ . Hence, the closed-loop dynamical system given by (12), (15), and (16) is Lyapunov stable for all  $(0, \tilde{w}_{i0}, \tilde{w}_{ij0})$ ,  $t \in \overline{\mathbb{R}}_+$ , and  $(i, j)$ .

Next, it follows from [20] that  $\lim_{t \rightarrow \infty} e^T(t)(\mathcal{L}(\mathcal{G}) + K)e(t) = 0$  holds, which implies that  $\lim_{t \rightarrow \infty} e(t) = 0$  as a consequence of  $\det(\mathcal{L}(\mathcal{G}) + K) \neq 0$ . The result  $\lim_{t \rightarrow \infty} x(t) = (\mathbf{1}_n \mathbf{1}_n^T / n)x_0$  is now immediate. ■

**Remark 5.** We now revisit the example in Remark 1 and use the adaptive control input given by (8) with the update laws given by (9) and (10). In particular, we set  $(k_1, k_2, k_3) = (5, 5, 0)$  and  $\gamma_i = \gamma_{ij} = 5$  for all  $(i, j)$ , and use zero initial conditions for the update laws. Figure 2 shows that the proposed approach achieves average consensus for all cases, i.e.,  $\lim_{t \rightarrow \infty} x_i(t) = 0.6$  for all  $i$ , as expected from Theorem 2.

## 5. Conclusion

In order to contribute to the previous studies in networked multiagent systems, we investigated an adaptive control methodology that has an ability to drive the trajectories of an uncertain and time-varying multiagent system to a close neighborhood of the trajectories of a given reference model having a desired graph topology. In the context of average consensus problem, we rigorously analyzed stability properties of this methodology using results from nonlinear systems theory and matrix mathematics. In addition, as a special case when the unknown coefficients of the graph are constant, we showed that the uncertain multiagent system asymptotically converges to the given reference model with the proposed control methodology. Illustrative examples indicated that the presented theory and its numerical results are compatible.



**Figure 2:** Trajectories of three agents on a line graph subject to initial conditions  $(x_{10}, x_{20}, x_{30}) = (0.2, 0.4, 1.2)$  and the proposed approach for (a)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 1, 1, 1)$ , (b)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1.1, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 0.1, 1, 1)$ , (c)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 2, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (-1, -1, 1, 1)$ , and (d)  $(\alpha_1, \alpha_2, \alpha_3) = (1, 1.5, 1)$  and  $(\beta_{12}, \beta_{21}, \beta_{23}, \beta_{32}) = (1, 1, 1, 1)$ .

## References

- [1] M. Mesbahi and M. Egerstedt, *Graph theoretic methods in multiagent networks*. Princeton University Press, 2010.
- [2] S. Sundaram and C. N. Hadjicostis, “Distributed function calculation via linear iterations in the presence of malicious agents: Attacking the network,” *American Control Conference*, pp. 1350–1355, 2008.
- [3] —, “Distributed function calculation via linear iterations in the presence of malicious agents: Overcoming malicious behavior,” *American Control Conference*, pp. 1356–1361, 2008.
- [4] M. Franceschelli, M. Egerstedt, and A. Giua, “Motion probes for fault detection and recovery in networked control systems,” *American Control Conference*, pp. 4358–4363, 2008.
- [5] F. Pasqualetti, A. Bicchi, and F. Bullo, “Distributed intrusion detection for secure consensus computations,” *IEEE Conference on Decision and Control*, pp. 5594–5599, 2007.
- [6] —, “Consensus computation in unreliable networks: A system theoretic approach,” *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 90–104, 2012.
- [7] I. Shames, A. M. Teixeira, H. Sandberg, and K. H. Johansson, “Distributed fault detection for interconnected second-order systems,” *Automatica*, vol. 47, no. 12, pp. 2757–2764, 2011.
- [8] T. Yucelen and M. Egerstedt, “Control of multiagent systems under persistent disturbances,” *American Control Conference*, pp. 5264–5269, 2012.
- [9] —, “Distributed control of multiagent systems under unknown persistent disturbances,” *arXiv preprint arXiv:1310.3489*, 2013.
- [10] M. Fazlyab and V. M. Preciado, “Robust topology identification and control of LTI networks,” *arXiv preprint arXiv:1406.4201*, 2014.
- [11] J. Zhou and J.-a. Lu, “Topology identification of weighted complex dynamical networks,” *Physica A: Statistical Mechanics and Its Applications*, vol. 386, no. 1, pp. 481–491, 2007.
- [12] H. Tang, L. Chen, J.-a. Lu, and C. K. Tse, “Adaptive synchronization between two complex networks with nonidentical topological structures,” *Physica A: Statistical Mechanics and its Applications*, vol. 387, no. 22, pp. 5623–5630, 2008.
- [13] Y. Xu, W. Zhou, J. Fang, and H. Lu, “Structure identification and adaptive synchronization of uncertain general complex dynamical networks,” *Physics Letters A*, vol. 374, no. 2, pp. 272–278, 2009.

- [14] C. Altafini, "Consensus problems on networks with antagonistic interactions," *Automatic Control, IEEE Transactions on*, vol. 58, no. 4, pp. 935–946, 2013.
- [15] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *Automatic Control, IEEE Transactions on*, vol. 49, no. 9, pp. 1520–1533, 2004.
- [16] J.-B. Pomet and L. Praly, "Adaptive nonlinear regulation: estimation from the lyapunov equation," *Automatic Control, IEEE Transactions on*, vol. 37, no. 6, pp. 729–740, 1992.
- [17] T. Yucelen and W. M. Haddad, "A robust adaptive control architecture for disturbance rejection and uncertainty suppression with  $\mathcal{L}_\infty$  transient and steady-state performance guarantees," *International Journal of Adaptive Control and Signal Processing*, vol. 26, no. 11, pp. 1024–1055, 2012.
- [18] D. C. Lay, *Linear Algebra And Its Applications*, 3rd ed. Boston, MA: Pearson, 2006.
- [19] D. S. Bernstein, *Matrix mathematics: theory, facts, and formulas*. Princeton University Press, 2009.
- [20] W. M. Haddad and V. Chellaboina, *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, 2008.